PLANE CURVES WITH CONSECUTIVE DOUBLE POINTS.

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The problem of the existence of plane curves of given order and genus has been completely solved for the case in which all the singularities are ordinary double points and are distinct.* Guccia has raised the question† as to how many of these double points can be consecutive. The problem has already been solved for the quartic and the quintic.‡ In this paper a method is developed for determining a curve of order n with d consecutive double points. A lower limit for the maximum value of d is determined. The equation of the rational sextic with all its double points consecutive is given.

I. The General Problem.

1. In this section a method for determining the equation of a plane curve of order n having d consecutive double points will be deduced. Let the equation of the curve C_n be taken in the form

$$\sum_{k=0}^{n}\sum_{i=0}^{n-k} A_{ki}x^k y^i = 0, \tag{1}$$

where the coefficients A_{ki} are to be determined. Let the point $(0, a_0)$ be the double point and let the equation of one of the branches through this point be

$$y = \sum_{t=0}^{\infty} a_t x^t. \tag{2}$$

If this value of y is substituted in (1), then

$$A_p = 0, \quad p = 0,\ 1,\ 2,\ \cdots, \tag{3}$$

where A_p denotes the coefficients of x^p in the resulting identity in x.

2. To obtain the explicit form of A_p, write

$$y^i = \sum_{p=k}^{\infty} B_{i,\,p-k}x^{p-k},$$

where

$$B_{i,\,p-k} = (i!)\Sigma\frac{\Pi a_t^{i_t}}{\Pi(i_t!)}, \tag{4}$$

* See V. Snyder, "Construction of plane curves of given order and genus, having distinct double points," Bulletin Amer. Math. Soc., Series 2, vol. 15, 1908, pp. 1–4.

† See Circ. Mat. Palermo, Supplemento ai Rendiconti, vol. 5, 1910, p. 1.

‡ See Salmon, Higher Plane Curves.

in which the two products are taken with respect to i_t, $(i_t!) = 1$ if $i_t = 0$, and the summation extends over all solutions in positive integers and zero of

$$
(5) \qquad \begin{aligned} i_0 + i_1 + i_2 + \cdots + i_t + \cdots + i_{p-k} &= i, \\ i_1 + 2i_2 \cdots + ti_t + \cdots + (p-k)i_{p-k} &= p - k. \end{aligned}
$$

When this value of y^i is substituted in (1), the equations (3) become

$$
(6_p) \qquad A_p = \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} B_{i,\, p-k} = 0, \; p = 0, 1, 2, \cdots.
$$

In these equations $B_{i,q} = 0$ if $q < 0$. By assigning to p the successive values 0, 1, 2, $\cdots$, sufficient linear equations in the unknown coefficients A_{ki} will be obtained to completely determine them. The rest of the equations (6_p) will then determine the remaining coefficients a_t of (2), beginning with a certain one of them, in terms of the others.

3. To determine the additional equations of condition on the A_{ki}, in order that C_n shall have double points, requires a discussion of the equations (6_p). From (5) it follows that *a_t can enter only linearly in A_p if $2t > p$ unless $t > p$, in which case a_t does not enter into the value of A_p.*

4. If in (6_p), we take $2s > p > s$, then the part of the coefficient of a_{p-s} in A_p which contains no a_t with $t > s$ is

$$
(7) \qquad a_s \sum_{i=0}^{n} \frac{i!}{(i-2)!} a^{i-2} A_{0i} + T_s,
$$

where

$$
T_s \equiv \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} B'_{i,\, s-k},
$$

in which $B'_{i,\, s-k}$ has the same form as (4) except that the summation is taken so that

$$
i_0 + i_1 + i_2 + \cdots + i_{s-1} = i - 1,
$$

$$
i_1 + 2i_2 + \cdots + (s-1)i_{s-1} = s - k.
$$

If $p = 2s$, if follows from (4), (5) and (6) that the part of A_{2s} which contains no a_t which $t > s$ is

$$
(8) \qquad A_{2s}' \equiv a_s^2 \sum_{i=0}^{n} \frac{i!}{2!(i-2)!} a_0^{i-1} A_{0i} + a_s T_s + T_s',
$$

where T_s has the same significance as in (7) and

$$
T_s' \equiv \sum_{k=0}^{n} \sum_{i=0}^{n-k} A_{ki} B''_{i,\, 2s-k},
$$

in which the summation for $B''_{i,\,2s-k}$ is taken so that

$$i_0 + i_1 + i_2 + \cdots + i_{s-1} = i,$$
$$i_1 + 2i_2 + \cdots + (s-1)i_{s-1} = 2s - k.$$

If $s = 0$, (7) and (8) become respectively

$$(7') \qquad \sum_{i=0}^{n} i a_0^{i-1} A_{0i},$$

$$(8') \qquad A_0 \equiv \sum_{i=0}^{n} a^i A_{0i}.$$

5. The successive conditions that C_n shall have 1, 2, $\cdots$, d double points can now be written down. In doing this it is desirable, for the moment, to think of the A_{ki} in the equations (6_p) as being known and of the coefficients a_t as unknown. Since $(0, a_0)$ is a double point on C_n, it counts for two of the intersections of (1) and $x = 0$; that is $A_0 = 0$ must have a_0 as a double root. Hence

$$(9_0) \qquad \sum_{i=0}^{n} a_0^i A_{0i} = 0$$

and

$$(9') \qquad \sum_{i=0}^{n} i a_0^{i-1} A_{0i} = 0$$

hold simultaneously. From § 3, $(9_0')$ and $(7')$ it follows that *if C_n has one double point at $(0, a_0)$ A_p contains no a_t with $t > p - 1$.*

6. From the hypothesis that C_n has d consecutive double points at $(0, a_0)$ it follows that there is a second branch of the curve through this point, and that the equation of this branch is

$$(2') \qquad y' = \sum_{t=0}^{d-1} a_t x^t + \sum_{t=d}^{\infty} a_t' x^t.$$

The equations which determine the coefficients $a_1, a_2, \cdots, a_{d-1}$ must therefore have these coefficients as double roots. These equations are quadratic in the unknown and the following lemma will be used. *If $ax^2 + 2bx + c = 0$ has double roots, $ax + b = 0$ and $bx + c = 0$ are consistent, and conversely.*

7. From § 5 it follows that (6_1) no longer contains a_1. If $s = 1$, by using (8) and § 5 we reduce (6_2) to $A_2' = 0$. In order that C_n may have two consecutive double points at $(0, a_0)$, $A_2' = 0$ must have equal roots. Using the lemma of § 6, $A_2' = 0$ and the condition for equal roots are replaced by the equations

$$(9_2) \qquad a_1 \sum_{i=0}^{n} i(i-1) a_0^{i-1} A_{0i} + T_1 = 0,$$

$$(9_2\ ) \qquad a_1 T_1 + 2T_1' = 0.$$

If $s = 1$, by combining (9_2) and (7) we deduce that A_{p-1}, $p > 2$, no longer contains a_{p-1}. Hence from § 5, *if C_n has two consecutive double points, A_p contains no a_t with $t > p - 2 > 0$.*

8. It should here be noticed that the equations (9_0), $(9_0')$ and (6_1) must be satisfied in order that C_n may have one double point, and that the three equations (9_2), $(9_2')$ and (6_3) must also be satisfied to ensure two consecutive double points. Each additional consecutive double point requires three conditions.

9. The argument of § 7 may be repeated with $s = 2$. The conclusion is that if C_n has three consecutive double points, A_p contains no a_t with $t > p - 3 > 0$. If the argument is repeated a sufficient number of times, the equations of condition, in order that C_n take on the last of $s + 1$ consecutive double points, are

$$(9_{2s}) \qquad a_s \sum_{i=0}^{n} i(i-1)a_0^{i-1}A_{0i} + T_s = 0,$$

$$(9_{2s}') \qquad a_sT_s + 2T_s' = 0$$

and (6_{2s+1}). From (7) it follows that *if C_n has $s + 1$ consecutive double points, A_p will contain no a_t with $t > p - \overline{s+1} > 0$.*

10. The curve (1) will have d consecutive double points if the conditions given by equations (9_{2s}), $(9_{2s}')$ and (6_{2s+1}) with $s = 0, 1, 2, \cdots, d-1$ are satisfied. None of the equations of condition will contain a coefficient a_t with $t > s$. Each time a new double point is added, the equations of condition will contain the a_s corresponding to this double point linearly. These $3d$ equations of condition will be referred to collectively as the equations (E) and are equations for determining the $\frac{1}{2}n(n + 3)$ unknown coefficients A_{ki} of (1).

11. If $D \equiv 3d - \frac{1}{2}n(n + 3) > 0$, there are D more equations than there are non-homogeneous unknowns A_{ki}. The least value of n for which this inequality can hold is $n = 6$, with $d = 10$. There is no problem if $n < 6$ and hence there are at least *seven* consecutive double points if $n \geqq 6$. It is therefore no restriction to take the expansion (2) in the canonical form*

$$(10) \qquad y = \tfrac{1}{2}x^2 + x^5 + \sum_{t=7}^{\infty} a_t x^t.$$

The necessary choice of the coördinate system uses all the projective constants in the expansion (2). The equations (E) are now supposed to be specialized to conform to the expansion (10).

12. Since there is no higher singularity at the origin (a_0 is now zero) than a double point it is no restriction to assume $A_{01} = 1$. If $D > 0$,

* See E. J. Wilczynski, Projective Differential Geometry of Curves and Ruled Surfaces, p. 84.

there are more equations than unknowns A_{ki} in the system (E). For these equations to be consistent requires the vanishing of D linearly independent determinants of order $\frac{1}{2}(n+1)(n+2)$ of the coefficients of A_{ki}. These determinants are rational integral functions of the $d-7$ coefficients a_t, $t = 7, 8, \cdots, d-1$. Hence the number of these equations can not exceed the number of the coefficients a_t. Thus

$$D \leqq d - 7$$

and therefore

$$d \leqq \tfrac{1}{4}(n^2 + 3n - 14). \tag{11}$$

This inequality gives a maximum for the value of d unless the curve C_n thus determined has additional consecutive double points.

13. The D equations of § 12, connecting the coefficients a_t of (10) have a number of simultaneous solutions. In case a solution of these equations is such that (10) is the expansion of a curve of order less than $n/2$, this solution must be rejected since the curve (1) is then composite. If the expansion (10) which corresponds to a solution of these equations is that of a curve of order greater than $n/2$, the curve (1) can not be composite. When n is even, the equation (1) may represent a curve of order $n/2$ counted twice of which (10) is the expansion given by a solution of these D equations.

II. The Sextic Curves.

14. The upper limit for d given by (11) is too low to assure the existence of rational curves of order higher than six with all their double points consecutive. The curves of order six with ten consecutive double points will now be determined.

15. Instead of starting with the sextic in the form (1) a known* result will be specialized. If $c_3 = 0$, and $c_3' = 0$, define a pencil of cubic curves with eight consecutive basis points and if $c = 0$ is a sextic, which is not a quadratic function of the cubics, having these eight basis points for double points then

$$Ac_3^2 + Bc_3c_3' + C(c_3')^2 + Dc = 0$$

will represent ∞^3 sextics with eight consecutive double points, the coefficients being constants. The curve c will satisfy the necessary conditions if it is the product of a conic $c_2 = 0$ through the first five basis points of the pencil of cubics and a quartic $c_4 = 0$ which has the remaining three consecutive basis points of the pencil for double points and one of its branches intersecting the pencil in the first five consecutive basis points.

* See V. Snyder, "The involutorial birational transformation of the plane, of order 17," Amer. Jour. Math., vol. 33, 1908, pp. 327–336; see p. 328.

16. Let the expansion (10) be written in the form

(12) $$y = \tfrac{1}{2}x^2 + x^5 + Tx^7 + Ux^8 + Vx^9 + Wx^{10} + \cdots.$$

The pencil of cubics is defined by*

$$c_3 \equiv y - \tfrac{1}{2}x^2 - 2Ty(y - \tfrac{1}{2}x^2) - 4xy^2 = 0,$$

$$c_3' \equiv x(y - \tfrac{1}{2}x^2) - 8y^3 = 0;$$

the conic is

$$c_2 \equiv y - \tfrac{1}{2}x^2 = 0;$$

the quartic is

$$c_4 \equiv T(y - \tfrac{1}{2}x^2)^2 - 2xy(y - \tfrac{1}{2}x^2) + 16y^4 = 0;$$

and the sextics are

(13) $$Ac_3^2 + Bc_3c_3' + C(c_3')^2 + Dc_2c_4 = 0.$$

17. If the equations (E) of § 10 are formed for this sextic, it will be found that all are satisfied except those for which $s = 8$ and $s = 9$ respectively. These are

(14) $$2A(U - 4) + BT - D = 0;$$

(15) $$BT(U - 4) + 2CT^2 - D(U - 8) = 0;$$

(16) $$B(U - 4)(U - 6) + 2CT(U - 6) + DT^2 = 0;$$

(17) $$A(V - T^2) + B(U - 5) + CT = 0;$$

(18) $$B[(V - T^2)(U - 5) + T^2(U - 4)] + C[T(V - T^2) + (U - 6)^2 + 2T^3] + 4DT = 0;$$

(19) $$B[V(V - T^2) + TU(U - 4)] + C[2T^2U + 2V(U - 6)] + D[20 + 3T^3 + U(6 - U)] = 0.$$

18. From (15) and (16) is obtained, since $D \neq 0$,

(20) $$T^3 + (U - 6)(U - 8) = 0;$$

from (14), (15) and (17) follows

(21) $$A[T(V - T^2) - (U - 4)(U - 6)] + D(U - 7) = 0;$$

from (14), (15) and (18) comes

(22) $$A[(U - 4)^2(U - 6)^2 - T(V - T^2)(U - 4)(U - 6)] + D[VT(U - 7) + 3T^3 - 2(U - 6)^2] = 0;$$

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* See Wilczynski, loc. cit., p. 67. All four curves may be constructed by the method of part I, or by the method given by Salmon.

and (19) becomes

$$(23)\quad \begin{aligned} &A[2V(U-4)^2(U-6) - 2VT(V-T^2)(U-4)] \\ &\quad + D[VT(V-T^2) - 4V(U-6) - 2T^2U + 20T^2 + 3T^5] = 0. \end{aligned}$$

From (20), (21) and (22) is obtained

$$(24)\qquad VT = -\frac{(U-6)(U-8)^2}{U-7},$$

and (23) gives

$$(25)\qquad T^2[U(T^3+2) - (T^3+3)(T^3+4)] = 0.$$

The solution $T = 0$ is rejected since (15) would violate $D \neq 0$. The second factor of (25) combined with (20) gives

$$(26)\qquad T^6 + T^3 - 1 = 0.$$

In succession follow

$$(27)\qquad \begin{aligned} U &= 7 - T^3, \\ V &= 2T^2 + T^5, \\ B &= -A(5T^2 + 4T^5), \\ C &= A(5T + 6T^4), \\ D &= A(2 - 3T^3). \end{aligned}$$

The equation $A_{20} = 0$ becomes, after the usual reduction,

$$\begin{aligned} &A[W-6T-TU]^2+D[6UT^2-2UV+6V-2T^2] \\ &\quad +B[(2V-T^2)(W-6T-TU)-2(U^2-16)+VT(U-4)+UT(V-T^2)] \\ &\qquad +C[2(U-6)(W-6T-TU)+V^2-16TU+2TU^2+2T^2V-16T]=0, \end{aligned}$$

from which

$$(28)\qquad W = \frac{42T - 7T^4 \pm \sqrt{-39T^2 + 63T^5}}{2}.$$

19. *If T satisfies* (26) *the curves*

$$c_3^2 - (5T^2 + 4T^5)c_3c_3' + (5T + 6T^4)(c_3')^2 + (2 - 3T^3)c_2c_4 = 0$$

are rational sextics having ten consecutive double points. The two branches of each of these curves have the expansion (12) *where T, U, V, W have the values given by* (26), (27) *and* (28). *All other rational sextics having ten consecutive double points are projectively equivalent to these.*

20. From (14) and (15) the values of B and C may be substituted in (13) and there results *the curves*

$$A[Tc_3 - (U-4)c_3']^2 + D[Tc_3c_3' - 2(c_3')^2 + T^2c_2c_4) = 0,$$

where T and U satisfy (20), *have nine consecutive double points. If $A = 0$ there is a triple point at* (0, 0).

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